Fuzzy Set Theory in Islamic Perspective

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1. INTRODUCTION

Computers do not reason as brains do. They reason in clear steps with statements that are black or white. They reason with strings of 0s and 1s. They cannot reason with vague terms of common sense as in "Abu-Talhā is a waliullāh", "Ibn-e-Ubaid is sāleh", "Fārābī is muttaqī", or "He is momin". These fuzzy or gray facts are true only to some degree between 0 and 1 and they are false to some degree. Brains work with these fuzzy patterns with ease and computer may not work them at all. Fuzzy set theory and logic try to convert those gray facts in such a form that can be accepted by computers. It deals with such situations where information may be somewhat vague.

2. VAGUENESS

Consider a set 'S' of a class of students S = {a, b, c ... z} where a, b, c ... z are the students of the class. Suppose in that class, four objects that are used for writing purpose are fountain pen, ball-point, pencil and marker. Suppose it is asked by the students a, b and c to write a set W of those objects that are used in the class for writing purpose and a set V of those students (excluding a, b and c) who are virtuous(sāleh). The set W constructed by the students will be same where as there is almost zero % chance that the set V will be same. Set W will be same because it is the set of well-defined objects. On the other hand, the word "virtuous" is ill-defined and vague that causes the difference in opinion and the set V is therefore different for different student. The set V is known as "fuzzy set".

3. FUZZY SET THEORY

According to each student in section 2 above, the "degree of virtuousness" of any two members that belong to his set V, is not necessarily the same. Suppose according to the student 'a', set V is

V = {e, h, l, m, p, s, y}

Now it is not necessary that according to "a's" opinion, students e and h possess the same "degree of virtuousness". Student "a" may say that e is more virtuous than h. While "a" was forming his set " V, his mind inclined towards the "degree of virtuousness" for each student. Suppose he was agreed to include those students in set V whose "degree of virtuousness"

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according to the acceptance of his mind was greater than 50% or 0.5. So what was his approach while he was forming the set V.His mind roughly assigned the "degree of virtuousness" to each student from a close interval [0,1]. He discarded those students whose "degree of virtuousness" was less than or equal to 0.5 and accepted those for his set V whose degree was greater than 0.5. So the students e, h, l, m, p, s and y, according to his opinion were so much virtuous that their "degree of virtuousness" was greater than 0.5.This "degree of virtuousness" in "Fuzzy Set Theory" is termed as "degree of membership". The formation rule of set V according to "Fuzzy Set Theory" is slightly different from the formation rule of student 'a' in the sense that "Fuzzy Set Theory" does not discard the students whose "degree of membership" is less than or equal to 0.5. Instead, it includes the students d to z in the set V with their degree of membership. The students having zero degree of membership are not supposed to be a member of the set V but all other are supposed to be a member of the set V at least to some extent. Although the assignment of degree of membership varies from person to person, yet the approach is more realistic and more acceptable and is applied in the field of "Artificial Intelligence" and "Expert System". The facts, relations, judgments, opinions and heuristics contained with in the expert knowledge are most of the time Fuzzy and Uncertain and it is desire able for an expert system to get conclusion through proper reasoning from these knowledge. The fuzzy set theory and fuzzy logic can be very helpful for this purpose.

The assignment of degree of membership may be expressed as a rule. The rule is called "Membership Function". e.g., suppose someone wants to define the set of first ten natural numbers 'close to 6'. Let the membership of the fuzzy set is:

$$\mu_{\tilde{6}}(x) = \frac{4}{4+(x-6)^2} \quad \text{for } x \in \{1, 2, 3, \ldots, 10\}$$

The fuzzy set can be expressed as:

$$\tilde{6} = 0.14/1 + 0.2/2 + 0.3/3 + 0.5/4 + 0.8/5 + 1/6 + 0.8/7 + 0.5/8 + 0.3/9 + 0.2/10$$

It can be observed that

i) The membership function reduces the fuzziness of the fuzzy set
ii) The number 5 is as much close to 6 as 7 and the degree of membership for both is 0.8

Take another example (for continuous case). Suppose someone wants to represent the set of middle-aged person as a fuzzy set, define on the interval [0, 80]. Let the membership function be:

$$\mu_{\text{middle-aged}}(x) = \begin{cases} 0 & \text{for either } 0 \le x \le 20 \text{ or } 60 \le x \le 80 \\ (x-20)/15 & \text{for } 20 < x < 35 \\ (60-x)/15 & \text{for } 45 < x < 60 \\ 1 & \text{for } 35 \le x \le 45 \end{cases}$$

The function is expressed in figure below:

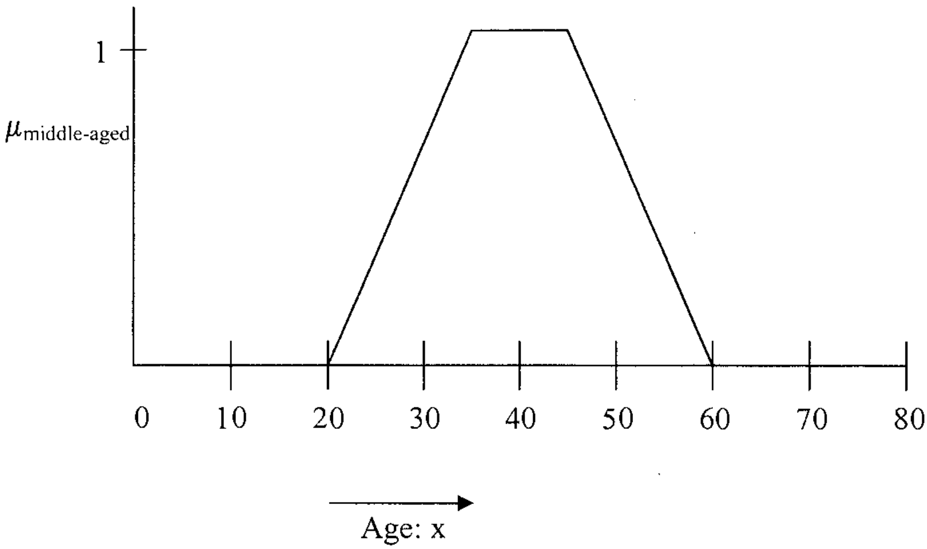

Figure 1: Fuzzy set of middle-aged persons

The example may be extended to three fuzzy sets that represent the concept of young, middle-aged and old persons.

$$\mu_{\text{young}}(x) = \begin{cases} 1 & \text{for } 0 \le x \le 20 \\ (35-x)/15 & \text{for } 20 < x < 35 \\ 0 & \text{for } 35 \le x \le 80 \end{cases}$$

$$\mu_{\text{middle-aged}}(x) = \begin{cases} 0 & \text{for either } 0 \le x \le 20 \text{ or } 60 \le x \le 80 \\ (x-20)/15 & \text{for } 20 < x < 35 \\ (60-x)/15 & \text{for } 45 < x < 60 \\ 1 & \text{for } 35 \le x \le 45 \end{cases}$$

$$\mu_{\text{old}}(x) = \begin{cases} 0 & \text{for } 0 \le x \le 45 \\ (x-45)/15 & \text{for } 45 < x < 60 \\ 1 & \text{for } 60 \le x \le 80 \end{cases}$$

These functions are expressed in the figure below:

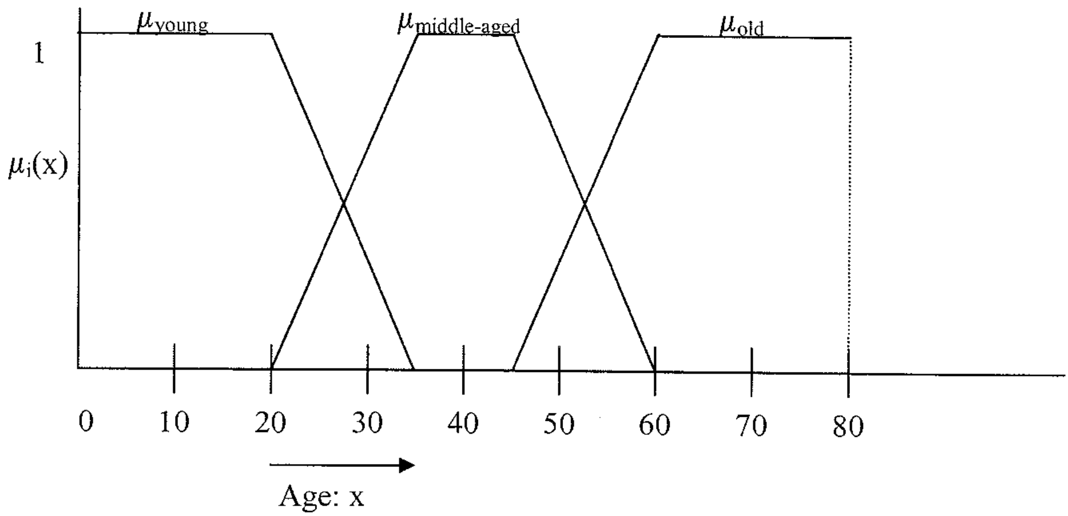

Figure 2: Fuzzy sets of young, middle-aged and old persons

The figure shows that μ_{young} is 1 in the interval [0, 20] whereas $\mu_{\text{middle-aged}}$ is zero in the interval. For the interval (20, 35), μ_{young} decreases whereas $\mu_{\text{middle-aged}}$ increases gradually. At $x = 35$, μ_{young} becomes zero whereas $\mu_{\text{middle-aged}}$ becomes 1. This is in accordance with the real phenomena.

4. CRISP SET THEORY VERSUS FUZZY SET THEORY

The basic assumption on which crisp set theory or classical set theory is based is that any object either belongs to or does not belong to a certain set A. A characteristic function of the set A assigns a value $\mu_A(x)$ to every x in the universe such that:

$$\mu_A(x) = \begin{cases} 0 \text{ if and only if } x \notin A \\ 1 \text{ if and only if } x \notin A \end{cases}$$

$$\text{i.e. } \mu_A: X \rightarrow \{0, 1\}$$

where X is the universal set.

The membership function discussed in section 3 is the generalized form of the characteristic function that assigns a value $\mu_A(x)$ to every x in the universe such that:

$$\mu_A: X \rightarrow [0, 1]$$

where [0, 1] denotes the closed interval from 0 to 1.

Consider the class discussed in section 2, that possesses four objects that are used for the writing purpose. Now suppose some other objects like table, chair, bag and duster are also present in the class. The crisp set of the objects that are use in the class for writing is:

W = {fountain pen, ball-point, pencil, marker}

The fuzzy set for objects that are used in the class for writing is:

T = {0/table, 0/chair, 0/bag, 0/duster, 1/fountain pen, 1/ ball-point, 1/pencil, 1/marker}

The two theories are agreed with each other in this situation and the two sets W and T are same.

Now consider the example discussed in section 3 once again. The three fuzzy sets that represent the young, middle-aged and old person are already represented and the figure 2 is also provided to elaborate the concept behind it. If the crisp set theory is used to represent the three sets then one has to use characteristic function instead of membership function. Now suppose one defines the following characteristic function.

$$\mu_{\text{young}}(x) = \begin{cases} 1 & \text{for } 0 \le x < 30 \\ 0 & \text{else where} \end{cases}$$

$$\mu_{\text{middle-aged}}(x) = \begin{cases} 1 \text{ for } 30 \le x < 60 \\ 0 \text{ else where} \end{cases}$$

$$\mu_{\text{old}}(x) = \begin{cases} 1 \text{ for } 60 \le x \le 80 \\ 0 \text{ else where} \end{cases}$$

These functions are expressed in the figure below where

(i) young is shown by the line............
(ii) middle-aged is shown by the line———
(iii) old is shown by the line-------

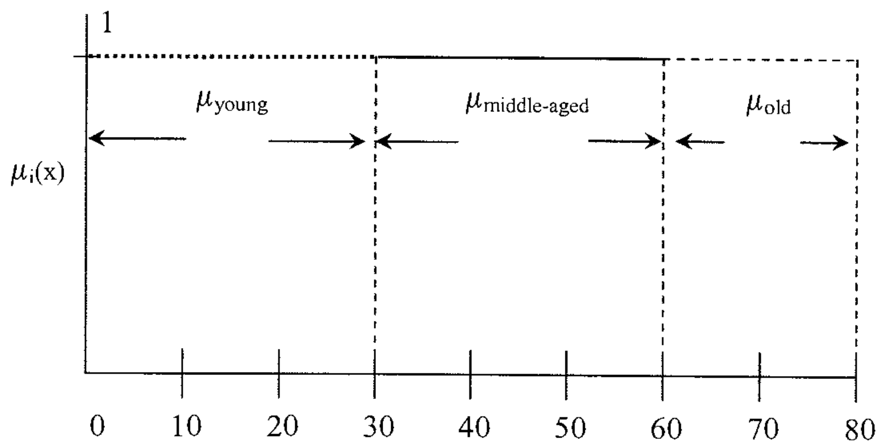

Figure 3: Crisp sets of young, middle-aged and old persons

The figure shows that the person is young in the interval [0, 30) and at x = 30, he is no more young and becomes middle-aged abruptly. He is middle-aged in the interval [30, 60) and at x = 60, he is no more middle-aged and becomes old abruptly. This is an unrealistic approach and is not in accordance with the real phenomena. On the other hand, as discussed in section 3, he is young with μ_{young} = 1 in the interval [0, 20]. The membership function μ_{young} gradually decreases after x = 20 and at the same time, the membership function $\mu_{middle\text{-}aged}$ starts increasing gradually. At x = 35, μ_{young} becomes zero whereas $\mu_{middle\text{-}aged}$ becomes 1. The person is middle-aged with $\mu_{middle\text{-}aged}$ = 1 in the interval [35,45]. $\mu_{middle\text{-}aged}$ gradually decreases after x = 45 and at the same time, μ_{old} starts increasing gradually. At x = 60, $\mu_{middle\text{-}aged}$ becomes 0 whereas μ_{old} becomes 1. He then remains old with μ_{old} = 1 up to the end. This is a realistic approach and is in accordance with the real phenomena. It is to be noted that the fuzzy set theory is a general form whereas crisp set theory is its particular type.

5. OPERATIONS ON FUZZY SETS

The three standard fuzzy operations are fuzzy complement, fuzzy intersection(t-norms) and fuzzy union(t-conorms). Other operations include the algebraic sum, disjunctive sum, product and absolute difference.

5.1. Fuzzy Complement

Let A be a fuzzy set with degree of membership $\mu_A(x)$, then Ā, the complement of the fuzzy set A is defined by a function

$$c: [0,1] \rightarrow [0,1]$$

which assigns a value $c(\mu_A(x))$ to each $x \in A$, where $c(\mu_A(x)) = 1 - \mu_A(x)$ e.g.

Let $X = \{x_1, x_2, x_3, x_4\}$ and $A = \{(0.1/x_1), (0.6/x_2), (0/x_3), (1/x_4)\}$ then

$\bar{A} = \{(0.9/x_1), (0.4/x_2), (1/x_3), (0/x_4)\}$

Diagrammatically, complement can be visualized as:

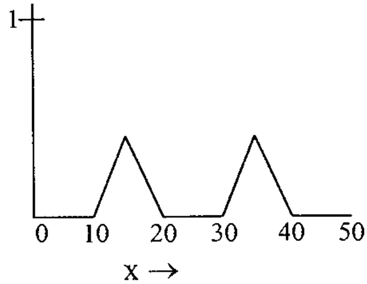

Figure 4: The set A(x)

then the complement $\bar{A}(x)$ is

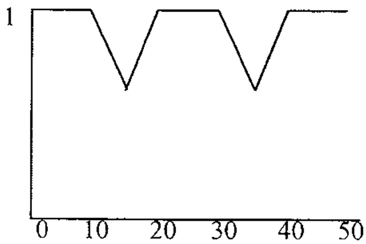

Figure 5: The set $\bar{A}(x)$

To produce meaningful fuzzy complements, function c must satisfy at least the following two axiomatic requirements

Axiom c1. $c(0) = 1$ and $c(1) = 0$ (boundary condition)
Axiom c2. For all a, b $\in [0,1]$, if $a \leq b$, then $c(a) \geq c(b)$ (monotonicity)

Consider the fuzzy complement

$$c(\mu_A(x)) = 1 - \mu_A(x)$$

For the axiom c1, $c(0) = 1 - 0 = 1$ and $c(1) = 1 - 1 = 0$

So axiom c1 is satisfied.

For the axiom c2, let a, b $\in [0, 1]$, where $a \leq b$, then $1 - a \geq 1 - b$ or $c(a) \geq c(b)$

So the axiom c2 is satisfied.

This shows that $c(\mu_A(x)) = 1 - \mu_A(x)$ is an acceptable fuzzy complement.

5.2. Fuzzy Intersection (t-norms)

Let A and B be two fuzzy sets with degrees of membership $\mu_A(x)$ and $\mu_B(x)$ then $A \cap B$, the intersection of the two sets is defined by a function

$$i: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

which assigns value $i[\mu_A(x), \mu_B(x)]$ to each $x \in A$ and $x \in B$, where,

$$i[\mu_A(x), \mu_B(x)] = \min[\mu_A(x), \mu_B(x)]$$

e.g.

Let X = {x1, x2, x3, x4}

$A = \{(0.1/x_1), (0.6/x_2), (0/x_3), (1/x_4)\}$ and

$B = \{(0.2/x_1), (0.4/x_2), (0.6/x_3), (0.8/x_4)\}$ then

$A \cap B = \{(0.1/x_1), (0.4/x_2), (0/x_3), (0.8/x_4)\}$

The following figure provides the intersection of two fuzzy sets A and B

(Note: The continuous line shows $\mu_{A \cap B}(x)$)

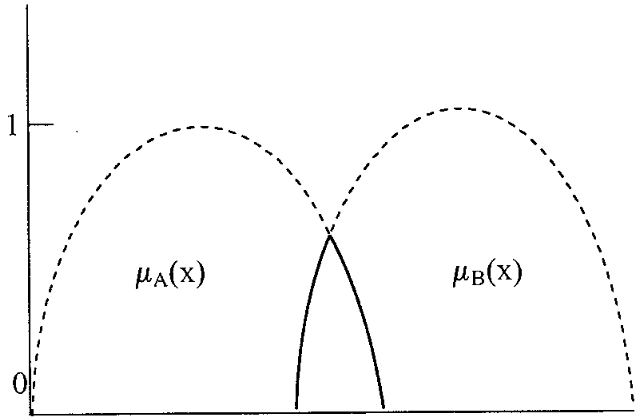

Figure 6: The intersection of two fuzzy sets.

A fuzzy intersection/t-norm i is a binary operation on the unit interval that satisfies at least the following four axioms for $a, b, c \in [0, 1]$.

Axiom i1. $i(a, 1) = a$ (boundary condition)

Axiom i2. $b \le c$ implies $i(a, b) \le i(a, c)$ (monotonicity)
Axiom i3. $i(a, b) = i(b, a)$ (commutativity)
Axiom i4. $i(a, i(b, c)) = i(i(a, b), c)$ (associativity)

Consider the fuzzy t-norm i

$$i[\mu_A(x), \mu_B(x)] = \min[\mu_A(x), \mu_B(x)]$$

For the Axiom i1, $i(a, 1) = \min(a, 1)$
Here a is either less than or equal to 1 and in each case minimum of a and 1 is a

$\therefore$ $i(a, 1) = a$.
so axiom i1 is satisfied

For the Axiom i2, let $b \le c$

The following 8 cases then arise

Case 1:

$a < b = c$.

In this case

$i(a, b) = \min(a, b) = a$ and
$i(a, c) = \min(a, c) = a$

and therefore $i(a, b) = i(a, c)$

Case 2:

$a = b = c =$ equal

In this case

$i(a, b) = \min(a, b) =$ equal and
$i(a, c) = \min(a, c) =$ equal

and therefore $i(a, b) = i(a, c)$

Case 3:

$b = c < a$

In this case

$i(a, b) = \min(a, b) = b$ and
$i(a, c) = \min(a, c) = c$

and therefore $i(a, b) = i(a, c)$

Case 4:

$$a < b < c$$

In this case

$$i(a, b) = \min(a, b) = a \text{ and}$$
$$i(a, c) = \min(a, c) = a$$

and therefore $i(a, b) = i(a, c)$

Case 5:

$$a = b < c$$

In this case

$$i(a, b) = \min(a, b) = a \text{ and}$$
$$i(a, c) = \min(a, c) = a$$

and therefore $i(a, b) = i(a, c)$

Case 6:

$$b < a < c$$

In this case

$$i(a, b) = \min(a, b) = b \text{ and}$$
$$i(a, c) = \min(a, c) = a$$

since $b < a$

therefore $i(a, b) < i(a, c)$

Case 7:

$$b < a = c$$

In this case

$$i(a, b) = \min(a, b) = b \text{ and}$$
$$i(a, c) = \min(a, c) = a$$

since $b < a$

therefore $i(a, b) < i(a, c)$

Case 8:

$$b < c < a$$

In this case

$$i(a, b) = \min(a, b) = b \text{ and}$$
$$i(a, c) = \min(a, c) = c$$

since $b < c$

therefore $i(a, b) < i(a, c)$

For the cases 1, 2, 3, 4 and 5, i(a, b) = i(a, c) and for the cases 6, 7 and 8, i(a, b) < i(a, c).
Thus $b \leq c$ implies $i(a, b) \leq i(a, c)$ and therefore axiom i2 is satisfied.

For the axiom i3, let $a \leq b$

$i(a, b) = \min(a, b) = a$ and
$i(b, a) = \min(b, a) = a$

and for $b \leq a$

$i(a, b) = \min(a, b) = b$ and
$i(b, a) = \min(b, a) = b$

Thus in each case

$i(a, b) = i(b, a)$

and therefore axiom i3 is satisfied.

For the axiom i4, the following three cases arise

Case 1

The three values a, b and c are same i.e. a = b = c = same

$\therefore$ i(a, i(b, c)) = i(i(a, b), c) = same

Case 2

(a) Two values are same whereas the third one is different and the same are smaller.
(b) Two values are same whereas the third one is different and the same are larger.

In each case

i(a, i(b, c)) = i(i(a, b),c) = smaller

Case 3

The three value are different and in this case

i(a,i(b, c)) = i(i(a, b),c) = least

Thus in each case, Axiom i4 holds.

The fuzzy intersection $i[\mu_A(x), \mu_B(x)] = \min [\mu_A(x), \mu_B(x)]$ satisfies the above four requirements (Axiom i1 to Axiom i4) and therefore it is an acceptable fuzzy intersection.

5.3. Fuzzy Union: (t-conorms)

Let A and B be two fuzzy sets with degrees of membership $\mu_A(x)$ and $\mu_B(x)$ then AUB, the union of the two sets is defined by a function.

$$u: [0,1]\times[0,1]\rightarrow[0,1]$$

which assigns value U $[\mu_A(x), \mu_B(x)]$ to each x, $x\in$ A and $x\in$ B, where

$$u[\mu_A(x), \mu_B(x)] = \max [\mu_A(x), \mu_B(x)] \text{ e.g.,}$$

Let

$A = \{ 0.1/ x_1, 0.6/ x_2, 0/ x_3, 1/ x_4 \}$ and

$B = \{ 0.2/ x_1, 0.4/ x_2, 0.6/ x_3, 0.8/ x_4 \}$ then

$AUB = \{ 0.2/ x_1, 0.6/ x_2, 0.6/ x_3, 1/ x_4 \}$

Following figure provide the union of two fuzzy sets A and B. (Note: the continuous line shows $\mu_{A\cup B}(x)$)

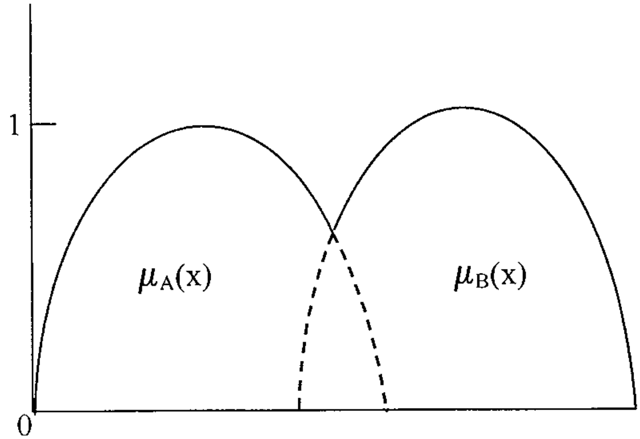

Figure 7: The union of two fuzzy sets.

A fuzzy union/t-conorms u is a binary operator on the unit interval that satisfies at least the following four axioms for a, b, c $\in$ [0,1]

Axiom u1. u(a.0) = a (boundary condition)
Axiom u2. $b \leq c$ implies $u(a, b) \leq u(a, c)$ (monotonicity)
Axiom u3. u(a, b) = u(b, a) (commutativity)
Axiom u4. u(a, u(b, c)) = u(u(a, b), c) (associativity)

Consider the fuzzy t-co norm

$$u[\mu_A(x), \mu_B(x)] = \max [\mu_A(x), \mu_B(x)]$$

For the axiom u1,

$u(a, 0) = \max(a, 0)$

Here a is either greater than or equal to zero and therefore the maximum of a and 0 is a.

$\therefore\ u(a, 0) = a$

So axiom u1 is satisfied

For the axiom u2, let $b \leq c$, then the following 8 cases arise

Case 1:

$a < b = c$
$u(a, b) = \max(a, b) = b$
$u(a, c) = \max(a, c) = c$
Since $b = c$
Therefore $u(a, b) = u(a, c)$

Case 2:

$a = b = c =$ equal
$u(a, b) = \max(a, b) =$ equal
$u(a, c) = \max(a, c) =$ equal
and therefore $u(a, b) = u(a, c)$

Case 3:

$b = c < a$
$u(a, b) = \max(a, b) = a$
$u(a, c) = \max(a, c) = a$
Therefore $u(a, b) = u(a, c)$

Case 4:

$a < b < c$
$u(a, b) = \max(a, b) = b$
$u(a, c) = \max(a, c) = c$
since $b < c$
therefore $u(a, b) < u(a, c)$

Case 5:

$a = b < c$
$u(a, b) = \max(a, b) = b$
$u(a, c) = \max(a, c) = c$
since $b < c$
therefore $u(a, b) < u(a, c)$

Case 6:

$b < a < c$
$u(a, b) = \max(a, b) = a$
$u(a, c) = \max(a, c) = c$
since $a < c$
therefore $u(a, b) < u(a, c)$

Case 7:

$b < a = c$
$u(a, b) = \max(a, b) = a$
$u(a, c) = \max(a, c) = a$
therefore $u(a, b) = u(a, c)$

Case 8:

$b < c < a$
$u(a, b) = \max(a, b)= a$
$u(a, c) = \max(a, c) = a$
therefore $u(a, b) = u(a, c)$

For the cases 1,2,3,7 and 8, $u(a, b) = u(a, c)$ and for the cases 4,5 and 6, $u(a, b) < u(a, c)$
Thus $b \leq c$ implies $u(a, b) \leq u(a, c)$ and therefore axiom u2 is satisfied.

For the axiom u3, let $a \leq b$, then
$u(a, b) = \max(a, b) = b$ and
$u (b, a) = \max(b, a) = b$

Now let $b \leq a$, then
$u(a, b) = \max(a, b) = a$, and
$u(b, a) = \max(b, a) = a$

Thus in each case

$$u(a, b) = u(b, a)$$

and therefore axiom u3 is satisfies

For the axiom u4, the following three cases arise

Case 1

The three values a, b and c are same i.e. a = b = c = same

$\therefore$ u(a, u(b, c)) = u(u(a, b), c) = same

Case 2

(a) Two values are same whereas the third one is different and the same are smaller.

(b) Two values are same whereas the third one is different and the same are larger.

In each case

u(a, u(b, c)) = u(u(a, b), c) = larger

Case 3

The three value are different and in this case

u(a, u(b, c)) = u(u(a, b), c) = largest

Thus in each case, axiom u4 holds.

The formula $u[\mu a(x), \mu b(x)] = \max [\mu a(x), \mu b(x)]$ satisfies the above four requirements (axiom u1 to axiom u4) therefore it is an acceptable fuzzy union.

5.4. Algebraic Sum

Let A and B be the two fuzzy sets with membership functions $\mu_A(x)$ and $\mu_B(x)$, defined over the universal set X.

The algebraic sum of A and B is denoted by $A \hat{+} B$ and defined as the fuzzy set with the membership function

$$\mu_{A \hat{+} B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\, \mu_B(x)$$ e.g.,

Let

$X = \{x_1, x_2, x_3, x_4\}$

$A = \{0.1/ x_1, 0.6/ x_2, 0/ x_3, 1/ x_4\}$ and

$B = \{0.2/ x_1, 0.4/ x_2, 0.6/ x_3, 0.8/ x_4\}$ then

$A \hat{+} B = \{0.28/ x_1, 0.76/ x_2, 0.6/ x_3, 1/ x_4\}$

5.5. Disjunctive Sum

Let A and B be the two fuzzy sets with membership functions $\mu_A(x)$ and $\mu_B(x)$ defined over the universal set X. The disjunctive sum of A and B is denoted by $A \oplus B$ and is defined as the fuzzy set with the membership function.

$$\mu_{A \oplus B}(x) = \max[\min(\mu_A(x), 1 - \mu_B(x)), \min(1 - \mu_A(x), \mu_B(x))]$$

Let $X = \{x_1, x_2, x_3, x_4\}$
$A = \{0.1/ x_1, 0.6/ x_2, 0/ x_3, 1/ x_4\}$ and
$B = \{0.2/ x_1, 0.4/ x_2, 0.6/ x_3, 0.8/ x_4\}$ then
$\mu_{A \oplus B}(x_1) = \max[\min(\mu_A(x_1), (1 - \mu_B(x_1)], \min(1 - \mu_A(x_1), \mu_B(x_1))] = \max[\min(0.1,0.8), \min(0.9,0.2)] = \max(0.1,0.2) = 0.2$
$\mu_{A \oplus B}(x_2) = \max[\min(0.6,0.6), \min(0.4,0.4)] = \max(0.6,0.4) = 0.6$
$\mu_{A \oplus B}(x_3) = \max[\min(0,0.4), \min(1,0.6)] = \max(0,0.6) = 0.6$
$\mu_{A \oplus B}(x_4) = \max[\min(1,0.2), \min(0,0.8)] = \max(0.2,0) = 0.2$
$\therefore A \oplus B = \{0.2/ x_1, 0.6/ x_2, 0.6/ x_3, 0.2/ x_4\}$

5.6. Product

Let A and B be the two fuzzy sets with membership functions $\mu_A(x)$ and $\mu_B(x)$ defined over the universal set X. The product of A and B is denoted by A·B and is defined as the fuzzy set with the membership function

$$\mu_{A \cdot B}(x) = \mu_A(x).\mu_B(x)$$

Let
$X = \{x1, x2, x3, x4\}$
$A = \{0.1/ x_1, 0.6/ x_2, 0/ x_3, 1/ x_4\}$ and
$B = \{0.2/ x_1, 0.4/ x_2, 0.6/ x_3, 0.8/ x_4\}$ then

$A \cdot B = \{0.02/x1, 0.24/x2, 0/x3, 0.8/x4\}$

5.7. Absolute difference

The absolute difference of two fuzzy sets A and B denoted by |A-B| and defined by the fuzzy set whose membership function is

$$\mu_{|A-B|}(x) = |\mu_A(x) - \mu B(x)|$$

Let
$X = \{x1, x2, x3, x4\}$
$A = \{0.1/ x_1, 0.6/ x_2, 0/ x_3, 1/ x_4\}$ and
$B = \{0.2/ x_1, 0.4/ x_2, 0.6/ x_3, 0.8/ x_4\}$ then
$|A - B| = \{0.1/x1, 0.2/x2, 0.6/x3, 0.2/x4\}$

6. PROPERTIES OF FUZZY SET OPERATIONS

Fundamental properties of crisp set operations are *Involution, Commutativity, Associativity, Distributivity, Idempotence, Absorption, Absorption by X(universal set) and ∅, Identity, Law of contradiction, Law of excluded middle, De Morgan's laws.*

These are fundamental laws that are frequently used in proof theory. Fundamental fuzzy set operations (complement, intersection and union) obey these laws except the "law of contradiction" and the "law of excluded middle".

According to Fuzzy set theory, $A \cap \bar{A} \neq \varnothing$ and $A \cup \bar{A} \neq X$ in general. To verify, for example, that the law of contradiction is violated for fuzzy sets, we need only to show that the equation

$$\min[\mu_A(x), 1 - \mu_A(x)] = 0$$

is violated for at least one $x \in X$. Similarly to verify that the law of excluded middle is violated for fuzzy sets, we need only to show that the equation

$$\max[\mu_A(x), 1 - \mu_A(x)] = 1$$

is violated for at least one $x \in X$.

For this, consider the following figure

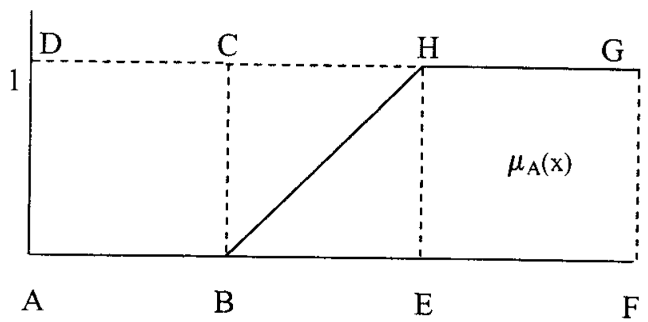

Figure 8: The set A(x).

Rectangles ABCD and EFGH are crisp regions, whereas rectangle BEHC is a non-crisp region. The complement of $\mu_A(x)$ i.e. $1 - \mu_A(x)$ is given below:

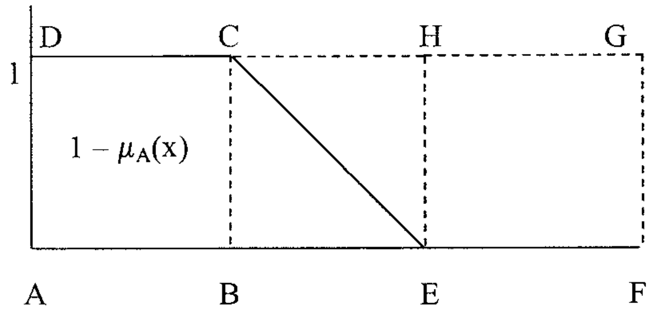

Figure 9: The set Ā(x)

Combining these two figures.
(Note: the continuous line shows $A \cup \bar{A}$ and dotted line (········) shows $A \cap \bar{A}$)

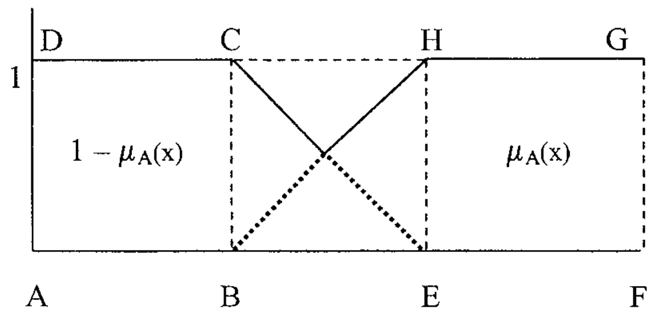

Figure 10: The sets A(x) and Ā(x).

The complete fuzzy region can be divided into (i) crisp region (ii) non-crisp region. In figure 10 rectangles ABCD and EFGH are crisp regions whereas region BEHC is a non-crisp region. It can be seen in the figure clearly that in non-crisp region, $\min[\mu_A(x), 1- \mu_A(x)] \neq 0$ and $\max[\mu_A(x), 1- \mu_A(x)] \neq 1$ which shows that the "law of contradiction" and the "law of excluded middle" do not hold in the non-crisp region.

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